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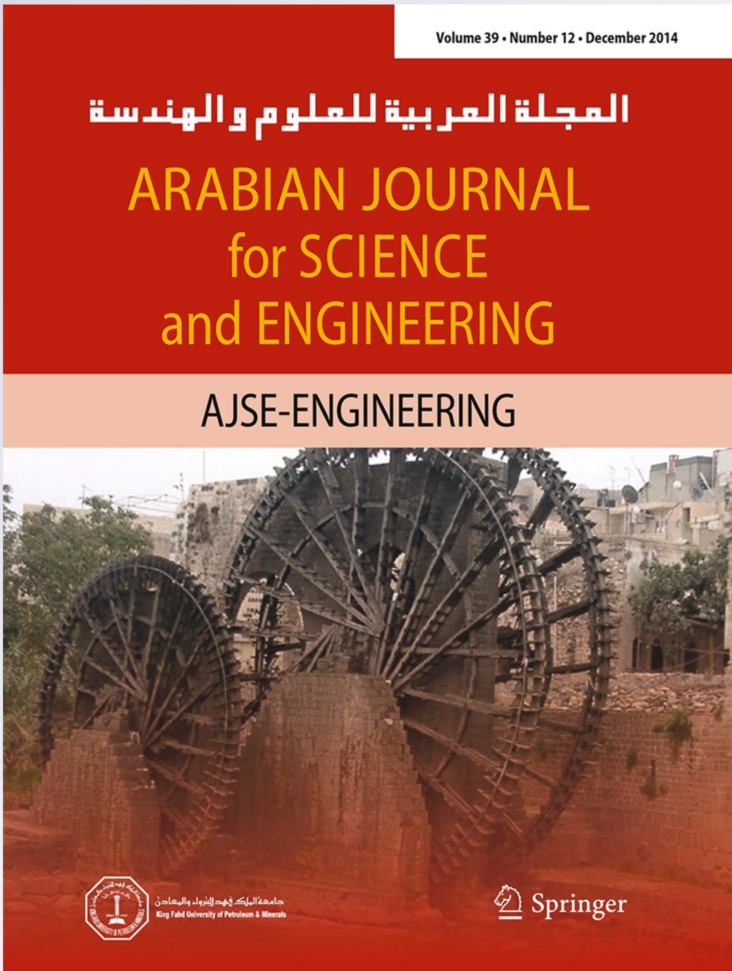

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Adomian Decomposition Method (ADM) Versus Nonlinear Solution of Natural Gas Compressibility Factor Correlations

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Abstract Many mathematical correlations were proposed for the calculation of the natural gas compressibility or z factor. Those correlations are implicit in z which means that they are iteratively solved using nonlinear solvers. In some normal cases, however, nonlinear solvers may diverge from the solution. This behavior has led some researchers to think about simpler and explicit methods for solving the z factor correlations. In a recent paper by Fatoorehchi et al. (J Petrol Sci Eng. doi:10.1016/j.petrol.2014.03.004, 2004), the authors have implemented the *Adomian Decomposition Method* (ADM) which was introduced by George Adomian in the 1980's. They used the ADM method to tackle the nonlinearity of natural gas z-factor of the Hankinson–Thomas–Phillips correlation. The authors presented examples of gas gravities in the range 0.6–0.8 and pressure ranges no more than 6,000 psi. The contribution of this paper, however, is that it presents the discretization and solution method of the widely accepted natural gas z-factor correlations using the ADM method. It extensively covers the procedure and methodology of evaluating the Adomian polynomials along with a listing of the first ten polynomials and nonlinearities in a very compact form. In addition, it compares the nonlinear and ADM solutions via extensive examples for gas gravities 0.6–1.5 and for pressures 100–9,000 psi.

Keywords Gas · Compressibility factor · Gas Z-factor · Nonlinear solvers · Bisection-Newton–Raphson · Adomian Decomposition Method · ADM

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الخلاصة

استحدث الباحثون في مجال هندسة البترول كثيراً من المعادلات الرياضية لحساب معامل حيود الغاز الذي يُعدُّ المطلب الأول لحساب خصائص الغاز لمكامن البترول. وأكثر هذه المعادلات كفاءةً، على كل حال ، هي معادلات ضمنية أو غير خطية. وبناءً على ذلك تستخدم الطرق الرياضية غير الخطية لحل هذه المعادلات ، مثل طريقة نيوتن-رافسون التي تُعدُّ الأشهر والأكثر تطبيقاً. وحيث إن الطرق الرياضية غير الخطية، في بعض الظروف، لاتقترب من الحل الصحيح، مما دفع بعض الباحثين إلى البحث عن طرق أخرى مباشرة لحل هذه المعادلات. وفي الثمانينيات الميلادية قدم البروفيسور جورج أدوميان طريقة رياضية تجزيئية لحل المعادلات غير الخطية ، ومنذ ذلك الحين عرفت بطريقة أدوميان التجزيئية (ADM). وقدم في العام الحالي الباحث فاتورتشي وسبعةٌ آخرون معه بحثاً استخدم فيه طريقة أدوميان التجزيئية لحل معادلة هانكنسن - توماس – فيليب ، حيث استخدم في بحثه متسلسلة أدوميان لحساب معامل حيود الغاز عند الظروف الطبيعية من الوزن الجزيئي للغاز (0.6 – 0.8) والضغط (أقل من 6000 بي إس آي) باستخدام ستة حدود في متسلسلة أدوميان. وتُعدُّ هذه الورقة من أوائل الأوراق العلمية التي تناقش طريقة حل المعادلات الرياضية لحساب معامل حيود الغاز الأكثر شيوعاً في أوساط صناعة البترول بطريقة أدوميان التجزيئية ، حيث تم استعراض حلول جميع الطرق الضمنية بهذه الطريقة. وتبين من هذا البحث أن طريقة أدوميان التجزيئية تمكنت من الحل فقط عند الظروف الطبيعية من الوزن الجزيئي للغاز (0.6 – 1.0) والضغط (أقل من 6000 بي إس آي) باستخدام عشرة حدود في متسلسلة أدوميان. ولو زادت قيم الوزن الجزيئي للغاز والضغط، فإنه يلزم زيادة عدد كثيرات الحدود في متسلسلة أدوميان لتصل إلى الثلاثين حداً ، حيث تظهر هذه الطريقة تذبذباً واضحاً مع زيادة الوزن الجزيئي للغاز والضغط، ويتبين ذلك من خلال الأمثلة المقدمة في البحث.

List of symbols

p	System pressure (psi)
p_{pc}	Pseudo-critical pressure of the gas mixture (psi)
$p_{pr} = \frac{p}{p_{pc}}$	Pseudo-reduced pressure of the gas mixture
T	System temperature (°R)

T_{pc} Pseudo-critical temperature of the gas mixture (°R)
$T_{pr} = \frac{T}{T_{pc}}$ Pseudo-reduced temperature of the gas mixture
z Compressibility (deviation) factor of the gas mixture

1 Introduction

The physical properties of petroleum gas mixtures such as the gas formation volume factor, density, solution gas–oil ratio, and viscosity are required for studying the future performance of petroleum reservoirs. The natural gas compressibility factor, however, must be evaluated first to be able to calculate other gas properties. The Standing–Katz [1] z-factor chart is the widely accepted chart as a practical source for the evaluation of the natural gas compressibility factor. Consequently, many correlations for evaluating the gas z-factor were proposed over the years. The rigorous and more efficient correlations, however, are nonlinear in nature such as Dranchuk–Abu-Kassem [2], Dranchuk–Purvis–Robinson [3], and Hankinson–Thomas–Phillips [4]. Nonlinear solvers are used for the solution of such correlations. The popular nonlinear solver is the Newton–Raphson method or simply Newton's method of solution. Other nonlinear solvers exist which may be preferable in certain situations. In some normal cases, these solvers do not converge to a solution.

In the 1980's, George Adomian (1923–1996) introduced a decomposition method for solving nonlinear equations. Since then, the method is known as the *Adomian Decomposition Method* (ADM) [5,6]. The method continues to develop and earn popularity in solving nonlinear problems. It is a reliable tool for solving nonlinear equations, providing in many cases the exact solution. More recently, some researchers [7] implemented the ADM method, for solving the Hankinson–Thomas–Phillips gas compressibility correlation. In this research, we discretize the above gas compressibility correlations and identify nonlinearities for the ADM method. Further we implement the method for the above-mentioned z-factor correlations at high gas gravities (0.6–1.5) and high pressures (greater than 6,000 psi). The complexity of the ADM method relies on the difficulty in the evaluation process of the *Adomian Polynomials*. In this paper, we present three published algorithms that we found in the literature and we believed adequate for the calculation of the Adomian polynomials. Such algorithms require the use of some symbolic software such as Maple, MATLAB, and Mathematica. To check the validity of both nonlinear and ADM solutions of the above correlations, we solved the correlations for gas gravities between 0.65 and 1.5 at a step size of 0.1 and for pressure values 100–9,000 psi at a step size of 100 psi.

Table 1 Constants of the Dranchuk–Abu-Kassem correlation

Coefficient	Value
a_0	0.32650
a_1	−1.07000
a_2	−0.53390
a_3	0.01569
a_4	−0.05165
a_5	0.54750
a_6	−0.73610
a_7	0.18440
a_8	−0.10560
a_9	0.61340
a_{10}	0.72100

2 Dranchuk–Abu-Kassem Correlation

Dranchuk and Abu-Kassem [2] proposed an eleven-constant correlation for evaluating the natural gas compressibility factor z. Their expression is written as follows:

$$z = 1 + x_0 x_4 \frac{1}{z} + x_1 x_4^2 \frac{1}{z^2} + x_2 x_4^5 \frac{1}{z^5} + x_3 x_4^2 \frac{1}{z^2} \exp\left(\frac{-a_{10} x_4^2}{z^2}\right) + a_{10} x_3 x_4^4 \frac{1}{z^4} \exp\left(\frac{-a_{10} x_4^2}{z^2}\right) \tag{1}$$

The equation coefficients $x_0 - x_4$ are given by:

$$\begin{aligned} x_0 &= a_0 + a_1 t + a_2 t^3 + a_3 t^4 + a_4 t^5, \\ x_1 &= a_5 + a_6 t + a_7 t^2, \\ x_2 &= a_8 \left(a_6 t + a_7 t^2\right), \\ x_3 &= a_9 t^3, \\ x_4 &= 0.27\, p_{pr}\, t, \text{ and} \\ t &= \frac{1}{T_{pr}} \end{aligned} \tag{2}$$

Values of the constants $a_0 - a_{10}$ are listed in Table 1.

3 Dranchuk–Purvis–Robinson Correlation

Dranchuk, Purvis, and Robinson [3] proposed an eight-constant correlation for evaluating the natural gas compressibility factor z. Their expression is written as follows:

Table 2 Constants of the Dranchuk–Purvis–Robinson correlation

Coefficient	Value
a_0	0.31506237
a_1	−1.04670990
a_2	−0.57832720
a_3	0.53530771
a_4	−0.61232032
a_5	−0.10488813
a_6	0.68157001
a_7	0.68446549

$$z = 1 + x_0 x_4 \frac{1}{z} + x_1 x_4^2 \frac{1}{z^2} + x_2 x_4^5 \frac{1}{z^5} + x_3 x_4^2 \frac{1}{z^2} \exp\left(\frac{-a_7 x_4^2}{z^2}\right) + a_7 x_3 x_4^4 \frac{1}{z^4} \exp\left(\frac{-a_7 x_4^2}{z^2}\right) \quad (3)$$

The equation coefficients $x_0 - x_4$ are given by:

$$x_0 = a_0 + a_1 t + a_2 t^3,$$
$$x_1 = a_3 + a_4 t,$$
$$x_2 = a_4 a_5 t,$$
$$x_3 = a_6 t^3,$$
$$x_4 = 0.27\, p_{\mathrm{pr}}\, t, \text{ and}$$
$$t = \frac{1}{T_{\mathrm{pr}}} \quad (4)$$

Values of the constants $a_0 - a_7$ are listed in Table 2.

4 Hankinson–Thomas–Phillips Correlation

Hankinson, Thomas, and Phillips [4] proposed an eight-constant correlation for calculating the natural gas compressibility factor z. Their expression is written as follows:

$$z = 1 + x_0 x_3 \frac{1}{z} + x_1 x_3^2 \frac{1}{z^2} + x_2 x_3^5 \frac{1}{z^5}\, exp\left(\frac{-a_7 x_3^2}{z^2}\right) + a_7 x_2 x_3^7 \frac{1}{z^7}\, exp\left(\frac{-a_7 x_3^2}{z^2}\right) \quad (5)$$

The equation coefficients $x_0 - x_3$ are given by:

$$x_0 = a_3 - a_1 t - a_5 t^3,$$
$$x_1 = a_2 - a_0 t,$$
$$x_2 = a_0 a_4 a_6 t,$$

Table 3 Constants of the Hankinson–Thomas–Phillips correlation

Coefficient	$0.4 < p_{\mathrm{pr}} < 5.0$	$5.0 \leq p_{\mathrm{pr}} < 15$
a_0	0.001290236	0.0014507882
a_1	0.381930050	0.3792226900
a_2	0.022199287	0.0241813990
a_3	0.122154810	0.1181228700
a_4	−0.015674794	0.0379056630
a_5	0.027271364	0.1984501600
a_6	0.023834219	0.0489116930
a_7	0.436177800	0.0631425417

$$x_3 = p_{\mathrm{pr}} t, \text{ and}$$
$$t = \frac{1}{T_{\mathrm{pr}}} \quad (6)$$

Values of the constants $a_0 - a_7$ are listed in Table 3.

5 Nonlinear Solution of the Gas Compressibility Factor Correlations

It is clear that the above correlations are nonlinear in the z factor; i.e., nonlinear solvers are required to solve them for the z factor, such as the Newton–Raphson method or simply Newton's method. This method is a popular technique for the solution of nonlinear equations, but alternative methods exist which may be preferable in certain situations. In some normal cases, however, such nonlinear solvers may diverge from the solution, unless fine-tuning precautions are implemented. In this research, we have implemented the Bisection nonlinear solver which is an improved root-finding scheme. It combines the bisection and Newton–Raphson methods and it requires two bounding points. The method, then, seeks a root that lies between these two bounds. The bisection method guarantees a root and is used to limit the changes in position estimated by the Newton–Raphson method when the linear assumption is poor. However, Newton–Raphson steps are taken in the nearly linear regime to speed-up convergence. In other words, if we know that we have a root bracketed between our two bounding points, we first consider the Newton–Raphson step. If that would predict a next point that is outside of our bracketed range, then we do a bisection step instead by choosing the midpoint of the range to be the next point. We then evaluate the function at the next point and, depending on the sign of that evaluation, replace one of the bounding points with the new point. This keeps the root bracketed, while allowing us to benefit from the speed of Newton–Raphson. This method was found the best to converge to a solution for the above gas compressibility factor correlations. The method was even tested using many rigorous and harsh conditions of gas composition, temperature, and pressure, and

Table 4 Adomian polynomials A_0-A_{10} listed for ease of access

$$A_0 = f(z_0)$$

$$A_1 = z_1 f^{(1)}(z_0)$$

$$A_2 = z_2 f^{(1)}(z_0) + \tfrac{1}{2!} z_1^2 f^{(2)}(z_0)$$

$$A_3 = z_3 f^{(1)}(z_0) + z_1 z_2 f^{(2)}(z_0) + \tfrac{1}{3!} z_1^3 f^{(3)}(z_0)$$

$$A_4 = z_4 f^{(1)}(z_0) + \left(z_1 z_3 + \tfrac{1}{2!} z_2^2\right) f^{(2)}(z_0) + \tfrac{1}{2!} z_1^2 z_2 f^{(3)}(z_0) + \tfrac{1}{4!} z_1^4 f^{(4)}(z_0)$$

$$A_5 = z_5 f^{(1)}(z_0) + (z_1 z_4 + z_2 z_3) f^{(2)}(z_0) + \tfrac{1}{2!}\left(z_1^2 z_3 + z_1 z_2^2\right) f^{(3)}(z_0) + \tfrac{1}{3!} z_1^3 z_2 f^{(4)}(z_0) + \tfrac{1}{5!} z_1^5 f^{(5)}(z_0)$$

$$\begin{aligned} A_6 = {} & z_6 f^{(1)}(z_0) + \left(z_1 z_5 + z_2 z_4 + \tfrac{1}{2!} z_3^2\right) f^{(2)}(z_0) + \left(\tfrac{1}{2!} z_1^2 z_4 + z_1 z_2 z_3 + \tfrac{1}{3!} z_2^3\right) f^{(3)}(z_0) + \left(\tfrac{1}{4} z_1^2 z_2^2 + \tfrac{1}{3!} z_1^3 z_3\right) f^{(4)}(z_0) \\ & + \tfrac{1}{4!} z_1^4 z_2 f^{(5)}(z_0) + \tfrac{1}{6!} z_1^6 f^{(6)}(z_0) \end{aligned}$$

$$\begin{aligned} A_7 = {} & z_7 f^{(1)}(z_0) + (z_3 z_4 + z_2 z_5 + z_1 z_6) f^{(2)}(z_0) + \left(\tfrac{1}{2!} z_2^2 z_3 + \tfrac{1}{2!} z_1 z_3^2 + z_1 z_2 z_4 + \tfrac{1}{2!} z_1^2 z_5\right) f^{(3)}(z_0) \\ & + \left(\tfrac{1}{3!} z_1 z_2^3 + \tfrac{1}{2!} z_1^2 z_2 z_3 + \tfrac{1}{3!} z_1^3 z_4\right) f^{(4)}(z_0) + \left(\tfrac{2}{4!} z_1^3 z_2^2 + \tfrac{1}{4!} z_1^4 z_3\right) f^{(5)}(z_0) + \tfrac{1}{5!} z_1^5 z_2 f^{(6)}(z_0) \\ & + \tfrac{1}{7!} z_1^7 f^{(7)}(z_0) \end{aligned}$$

$$\begin{aligned} A_8 = {} & z_8 f^{(1)}(z_0) + \left(\tfrac{1}{2!} z_4^2 + z_3 z_5 + z_2 z_6 + z_1 z_7\right) f^{(2)}(z_0) + \left(\tfrac{1}{2!} z_2 z_3^2 + \tfrac{1}{2!} z_2^2 z_4 + z_1 z_3 z_4 + z_1 z_2 z_5 + \tfrac{1}{2!} z_1^2 z_6\right) f^{(3)}(z_0) \\ & + \left(\tfrac{1}{4!} z_2^4 + \tfrac{1}{2!} z_1 z_2^2 z_3 + \tfrac{1}{4} z_1^2 z_3^2 + \tfrac{1}{2!} z_1^2 z_2 z_4 + \tfrac{1}{3!} z_1^3 z_5\right) f^{(4)}(z_0) \\ & + \left(\tfrac{2}{4!} z_1^2 z_2^3 + \tfrac{1}{3!} z_1^3 z_2 z_3 + \tfrac{1}{4!} z_1^4 z_4\right) f^{(5)}(z_0) + \left(\frac{1}{2(4!)} z_1^4 z_2^2 + \tfrac{1}{5!} z_1^5 z_3\right) f^{(6)}(z_0) + \tfrac{1}{6!} z_1^6 z_2 f^{(7)}(z_0) \\ & + \tfrac{1}{8!} z_1^8 f^{(8)}(z_0) \end{aligned}$$

$$\begin{aligned} A_9 = {} & z_9 f^{(1)}(z_0) + (z_4 z_5 + z_3 z_6 + z_2 z_7 + z_1 z_8) f^{(2)}(z_0) \\ & + \left(\tfrac{1}{3!} z_3^3 + z_2 z_3 z_4 + \tfrac{1}{2!} z_1 z_4^2 + \tfrac{1}{2!} z_2^2 z_5 + z_1 z_3 z_5 + z_1 z_2 z_6 + \tfrac{1}{2!} z_1^2 z_7\right) f^{(3)}(z_0) \\ & + \left(\tfrac{1}{3!} z_2^3 z_3 + \tfrac{1}{2!} z_1 z_2 z_3^2 + \tfrac{1}{2!} z_1 z_2^2 z_4 + \tfrac{1}{2!} z_1^2 z_3 z_4 + \tfrac{1}{2!} z_1^2 z_2 z_5 + \tfrac{1}{3!} z_1^3 z_6\right) f^{(4)}(z_0) \\ & + \left(\tfrac{1}{4!} z_1 z_2^4 + \tfrac{1}{4} z_1^2 z_2^2 z_3 + \tfrac{2}{4!} z_1^3 z_3^2 + \tfrac{1}{3!} z_1^3 z_2 z_4 + \tfrac{1}{4!} z_1^4 z_5\right) f^{(5)}(z_0) \\ & + \left(\tfrac{1}{3!3!} z_1^3 z_2^3 + \tfrac{1}{4!} z_1^4 z_2 z_3 + \tfrac{1}{5!} z_1^5 z_4\right) f^{(6)}(z_0) + \left(\frac{1}{2(5!)} z_1^5 z_2^2 + \tfrac{1}{6!} z_1^6 z_3\right) f^{(7)}(z_0) + \tfrac{1}{7!} z_1^7 z_2 f^{(8)}(z_0) \\ & + \tfrac{1}{9!} z_1^9 f^{(9)}(z_0) \end{aligned}$$

$$\begin{aligned} A_{10} = {} & z_{10} f^{(1)}(z_0) + \left(\tfrac{1}{2!} z_5^2 + z_4 z_6 + z_3 z_7 + z_2 z_8 + z_1 z_9\right) f^{(2)}(z_0) \\ & + \left(\tfrac{1}{2!} z_3^2 z_4 + \tfrac{1}{2!} z_2 z_4^2 + z_2 z_3 z_5 + z_1 z_4 z_5 + \tfrac{1}{2!} z_2^2 z_6 + z_1 z_3 z_6 + z_1 z_2 z_7 + \tfrac{1}{2!} z_1^2 z_8\right) f^{(3)}(z_0) \\ & + \left(\tfrac{1}{4} z_2^2 z_3^2 + \tfrac{1}{3!} z_1 z_3^3 + \tfrac{1}{3!} z_2^3 z_4 + z_1 z_2 z_3 z_4 + \tfrac{1}{4} z_1^2 z_4^2 + \tfrac{1}{2!} z_1 z_2^2 z_5 + \tfrac{1}{2!} z_1^2 z_3 z_5 + \tfrac{1}{2!} z_1^2 z_2 z_6 + \tfrac{1}{3!} z_1^3 z_7\right) f^{(4)}(z_0) \\ & + \left(\tfrac{1}{5!} z_2^5 + \tfrac{1}{3!} z_1 z_2^3 z_3 + \tfrac{1}{4} z_1^2 z_2 z_3^2 + \tfrac{1}{4} z_1^2 z_2^2 z_4 + \tfrac{1}{3!} z_1^3 z_3 z_4 + \tfrac{1}{3!} z_1^3 z_2 z_5 + \tfrac{1}{4!} z_1^4 z_6\right) f^{(5)}(z_0) \\ & + \left(\frac{1}{2(4!)} z_1^2 z_2^4 + \tfrac{2}{4!} z_1^3 z_2^2 z_3 + \frac{1}{2(4!)} z_1^4 z_3^2 + \tfrac{1}{4!} z_1^4 z_2 z_4 + \tfrac{1}{5!} z_1^5 z_5\right) f^{(6)}(z_0) \\ & + \left(\frac{1}{6(4!)} z_1^4 z_2^3 + \tfrac{1}{5!} z_1^5 z_2 z_3 + \tfrac{1}{6!} z_1^6 z_4\right) f^{(7)}(z_0) + \left(\frac{1}{2(6!)} z_1^6 z_2^2 + \tfrac{1}{7!} z_1^7 z_3\right) f^{(8)}(z_0) + \tfrac{1}{8!} z_1^8 z_2 f^{(9)}(z_0) \\ & + \tfrac{1}{10!} z_1^{10} f^{(10)}(z_0) \end{aligned}$$

They are calculated in a straightforward manner as follows: $A_i = A_i(z_0, z_1, \ldots, z_i) = \frac{1}{i!} \frac{d^i}{d\lambda^i} N\left(\sum_{k=0}^{\infty} z_k \lambda^k\right)\Big|_{\lambda=0}$

Table 5 Dranchuk–Abu-Kassem and Dranchuk–Purvis–Robinson first to fifth nonlinearities and their derivatives

Nonlinearity	Derivatives
$N(z_0) = f(z_0) = \dfrac{1}{z_0}$	$f^{(n)}(z_0) = (-1)^n \dfrac{n!}{z_0^{n+1}}, \quad n \geq 0$
$N(z_0) = f(z_0) = \dfrac{1}{z_0^2}$	$f^{(n)}(z_0) = (-1)^n \dfrac{(n+1)!}{z_0^{n+2}}, \quad n \geq 0$
$N(z_0) = f(z_0) = \dfrac{1}{z_0^5}$	$f^{(n)}(z_0) = (-1)^n \dfrac{(n+4)!}{z_0^{n+5}}, \quad n \geq 0$
$N(z_0) = f(z_0) = \frac{1}{z_0^2} \exp\left(\dfrac{-a_{10} x_4^2}{z_0^2}\right) = \dfrac{1}{z_0^2} exp\left(\dfrac{X}{z_0^2}\right)$	$f^{(n)}(z_0) = (-2)^n \left[\sum_{i=0}^{n} \left(\frac{C_{n,i} X^i}{z_0^{n+2+2i}}\right)\right] \exp\left(\dfrac{X}{z_0^2}\right), \quad n \geq 0$
$N(z_0) = f(z_0) = \frac{1}{z_0^4} \exp\left(\frac{-a_{10} x_4^2}{z_0^2}\right) = \dfrac{1}{z_0^4} exp\left(\frac{X}{z_0^2}\right)$	$f^{(n)}(z_0) = (-2)^n \left[\sum_{i=0}^{n} \left(\dfrac{D_{n,i} X^i}{z_0^{n+4+2i}}\right)\right] \exp\left(\frac{X}{z_0^2}\right), \quad n \geq 0$

Table 6 Constants $C_{n,i}$ for the Dranchuk–Abu-Kassem and Dranchuk–Purvis–Robinson fourth nonlinearity and its derivatives

1										
1	1									
1.5	3.5	1								
3	12	7.5	1							
7.5	45	45.75	13	1						
22.5	187.5	273.75	123.75	20	1					
78.75	866.25	1,693.125	1,078.125	273.75	28.5	1				
315	4,410	11,025	9,240	3,268.125	530.25	38.5	1			
1,417.5	24,570	76,072.5	80,325	37,019.0625	8,305.5	934.5	50	1		
7,087.5	148,837.5	557,077.5	718,672.5	413,496.5625	120,074.0625	18,585	1,534.5	63	1	
38,981.25	974,531.25	4,326,918.75	6,665,793.75	4,646,889.84375	1,674,274.21875	333,801.5625	37,766.25	2,385	77.5	1

Table 7 Constants $D_{n,i}$ for the Dranchuk–Abu-Kassem and Dranchuk–Purvis–Robinson fifth nonlinearity and its derivatives

1										
2	1									
5	5.5	1								
15	27	10.5	1							
52.5	136.5	84.75	17	1						
210	735	645	203.75	25	1					
945	4252.5	4927.5	2173.125	416.25	34.5	1				
4725	26460	38745	22312.5	5919.375	761.25	45.5	1			
25987.5	176715	317047.5	228401.25	78546.5625	13912.5	1284.5	58	1		
155925	1262992.5	2713095	2372658.75	1013866.875	231584.0625	29326.5	2038.5	72	1 1	
1013512.5	9628368.75	24324300	25253353.13	13018260.94	3677083.594	598165.3125	56846.25	3082.5	87.5	1

Table 8 Hankinson–Thomas–Phillips first to fourth nonlinearities and their derivatives

Nonlinearity	Derivatives
$N(z_0) = f(z_0) = \frac{1}{z_0}$	$f^{(n)}(z_0) = (-1)^n \frac{n!}{z_0^{n+1}}, \quad n \geq 0$
$N(z_0) = f(z_0) = \frac{1}{z_0^2}$	$f^{(n)}(z_0) = (-1)^n \frac{(n+1)!}{z_0^{n+2}}, \quad n \geq 0$
$N(z_0) = f(z_0) = \frac{1}{z_0^5}\exp\left(\frac{-a_7 x_3^2}{z_0^2}\right) = \frac{1}{z_0^5}\exp\left(\frac{X}{z_0^2}\right)$	$f^{(n)}(z_0) = (-2)^n \left[\sum_{i=0}^{n}\left(\frac{C_{n,i} X^i}{z_0^{n+5+2i}}\right)\right]\exp\left(\frac{X}{z_0^2}\right), \quad n \geq 0$
$N(z_0) = f(z_0) = \frac{1}{z_0^7}\exp\left(\frac{-a_7 x_3^2}{z_0^2}\right) = \frac{1}{z_0^7}\exp\left(\frac{X}{z_0^2}\right)$	$f^{(n)}(z_0) = (-2)^n \left[\sum_{i=0}^{n}\left(\frac{D_{n,i} X^i}{z_0^{n+7+2i}}\right)\right]\exp\left(\frac{X}{z_0^2}\right), \quad n \geq 0$

Table 9 Constants $C_{n,i}$ for the Hankinson–Thomas–Phillips third nonlinearity and its derivatives

1										
2.5	1									
7.5	6.5	1								
26.25	36.75	12	1							
105	210	108.75	19	1						
472.5	1,260	916.875	251.25	27.5	1					
2,362.5	8,032.5	7,678.125	2,926.875	498.75	37.5	1				
12,993.75	54,573.75	65,618.4375	32,556.5625	7,665	892.5	49	1			
77,962.5	395,010	579,521.25	358,627.5	109,206.5625	17,482.5	1,480.5	62	1		
506,756.25	3,040,537.5	5,320,940.625	3,986,482.5	1,505,296.40625	310,255.3125	35,988.75	2,317.5	76.5	1	
3,547,293.75	24,831,056.25	50,929,003.125	45,185,765.625	20,544,742.96875	5,228,360.15625	778,109.0625	68,433.75	3,465	92.5	1

Table 10 Constants $D_{n,i}$ for the Hankinson–Thomas–Phillips fourth nonlinearity and its derivatives

1										
3.5	1									
14	8.5	1								
63	60.75	15	1							
315	427.5	165.75	23	1						
1,732.5	3,093.75	1,670.625	361.25	32.5	1					
10,395	23,388.75	16,458.75	4,921.875	686.25	43.5	1				
67,567.5	185,810.625	163,288.125	63,216.5625	12,127.5	1,186.5	56	1			
472,972.5	1,554,052.5	1,655,403.75	795,453.75	196,619.0625	26,365.5	1,914.5	70	1		
3,547,293.75	13,682,418.75	17,280,388.125	10,007,668.125	3,056,572.96875	526,187.8125	52,211.25	2,929.5	85.5	1	
28,378,350	126,689,062.5	186,486,300	127,364,737.5	46,686,543.75	9,897,014.53125	1,257,145.3125	96,153.75	4,297.5	102.5	1

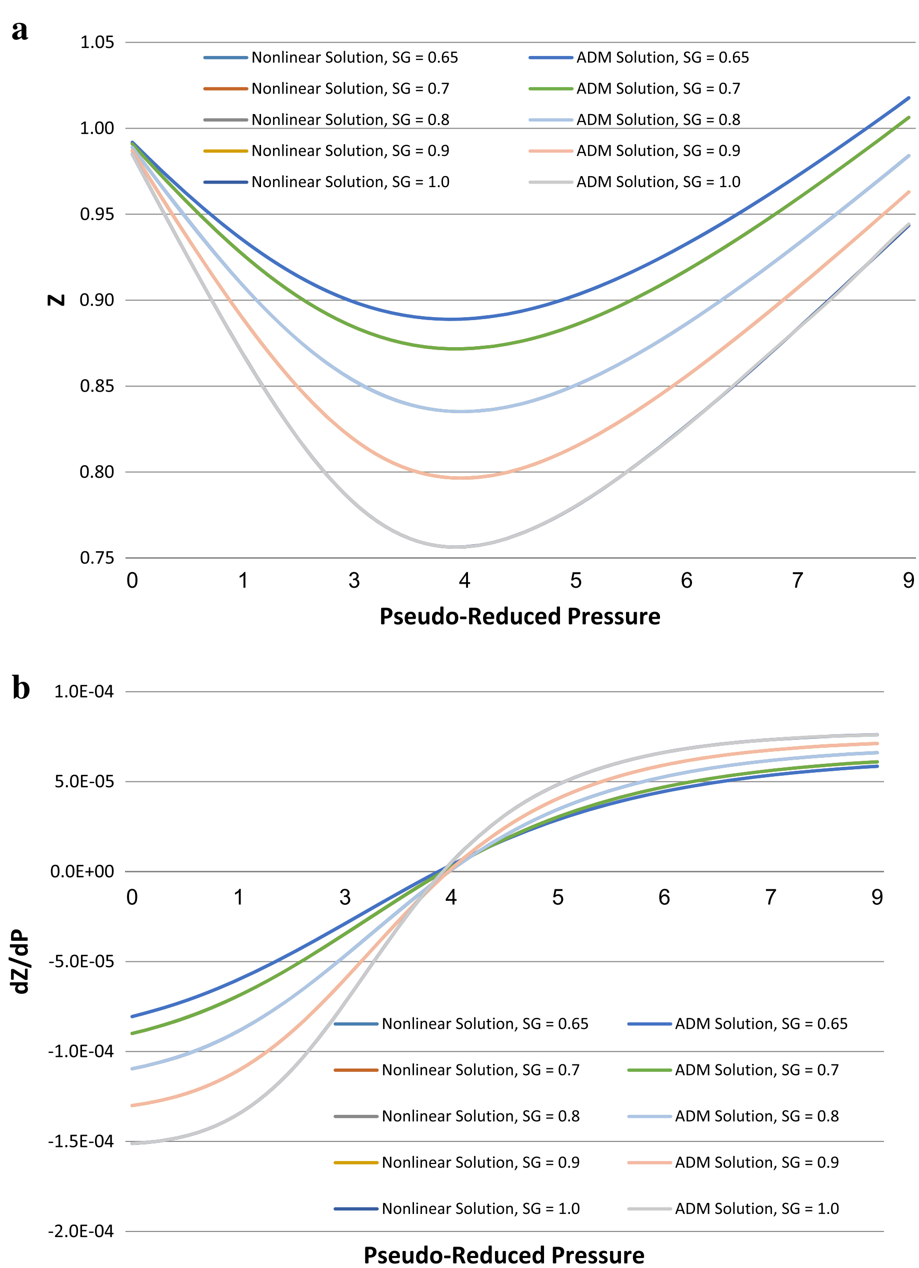

Fig. 1 **a** Plot of z versus pseudo-reduced pressure for nonlinear and ADM solutions, SG = 0.65–1.0, ADM 10-polynomials. **b** Plot of dzdp versus pseudo-reduced pressure for nonlinear and ADM solutions, SG = 0.65–1.0, ADM 10-polynomials

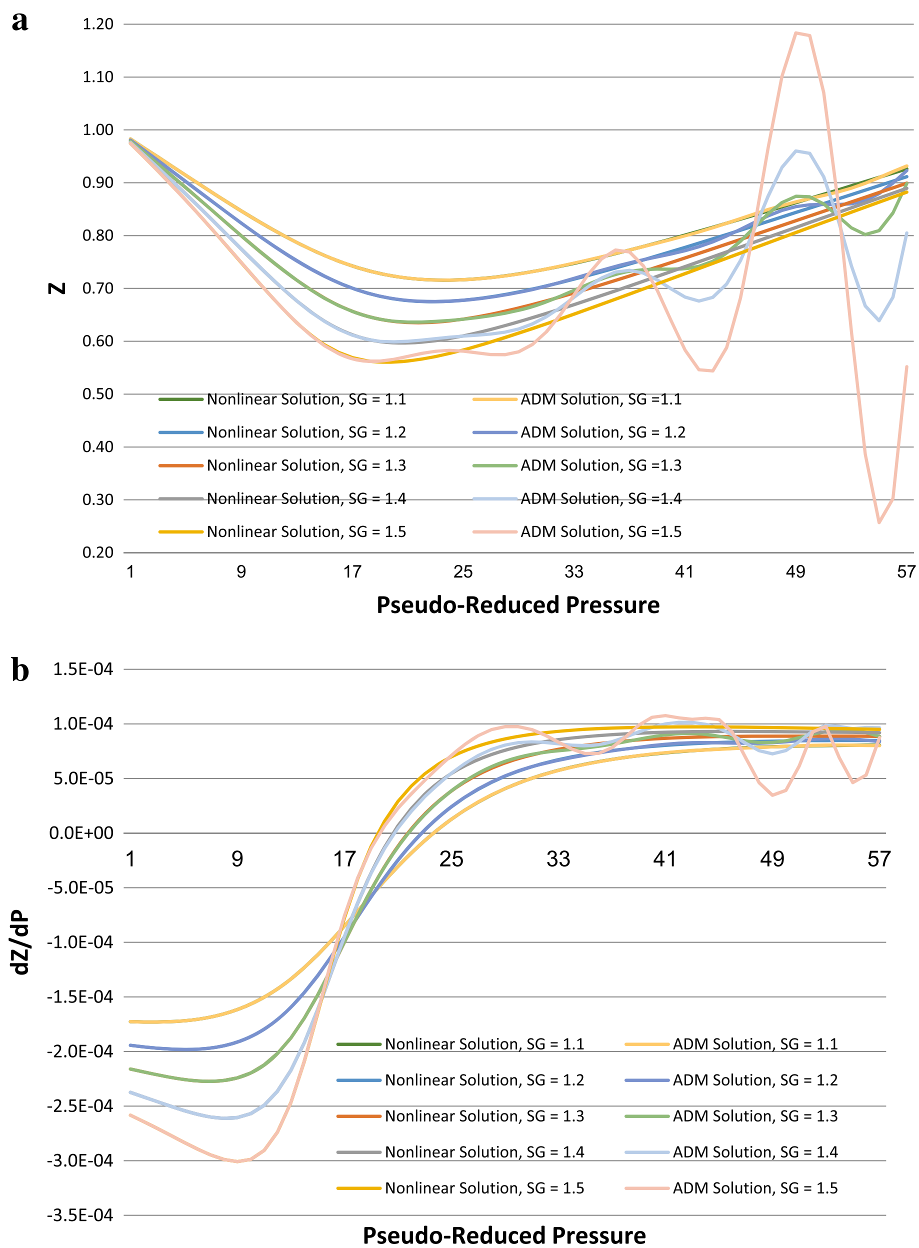

Fig. 2 **a** Plot of z versus pseudo-reduced pressure for nonlinear and ADM solutions, SG $= 1.1-1.5$, ADM 10-polynomials. **b** Plot of dzdp versus pseudo-reduced pressure for nonlinear and ADM solutions, SG $= 1.1-1.5$, ADM 10-polynomials

it did excellent. "Appendix A" is an efficient C# code of the Newton–Raphson-Bisection nonlinear solver.

6 ADM Solution of the Gas Compressibility Factor Correlations

In the 1980's, George Adomian (1923–1996) introduced a decomposition method for solving nonlinear equations. Since then, the method is known as the *Adomian Decomposition Method* (ADM) [5,6]. The method continues to develop and earn popularity in solving nonlinear problems. It is a reliable tool for solving nonlinear equations, providing in many cases the exact solution. In a recent paper by Fatoorehchi et al. [7], the authors have implemented this decomposition technique to tackle the nonlinearity of natural gas z-factor of the Hankinson–Thomas–Phillips correlation. The authors, however, did not deal with extremities such as gas gravities above 1.0 and high pressure ranges above 6,000 psi. In this research, we discretize the above correlations and identify nonlinearities for the ADM method. Further we implement the ADM method at high gas gravities (0.6–1.5) and high pressures (above 6,000 psi). The ADM method defines z, as a sum of components, by the following series:

$$z = \sum_{k=0}^{\infty} z_k \tag{7}$$

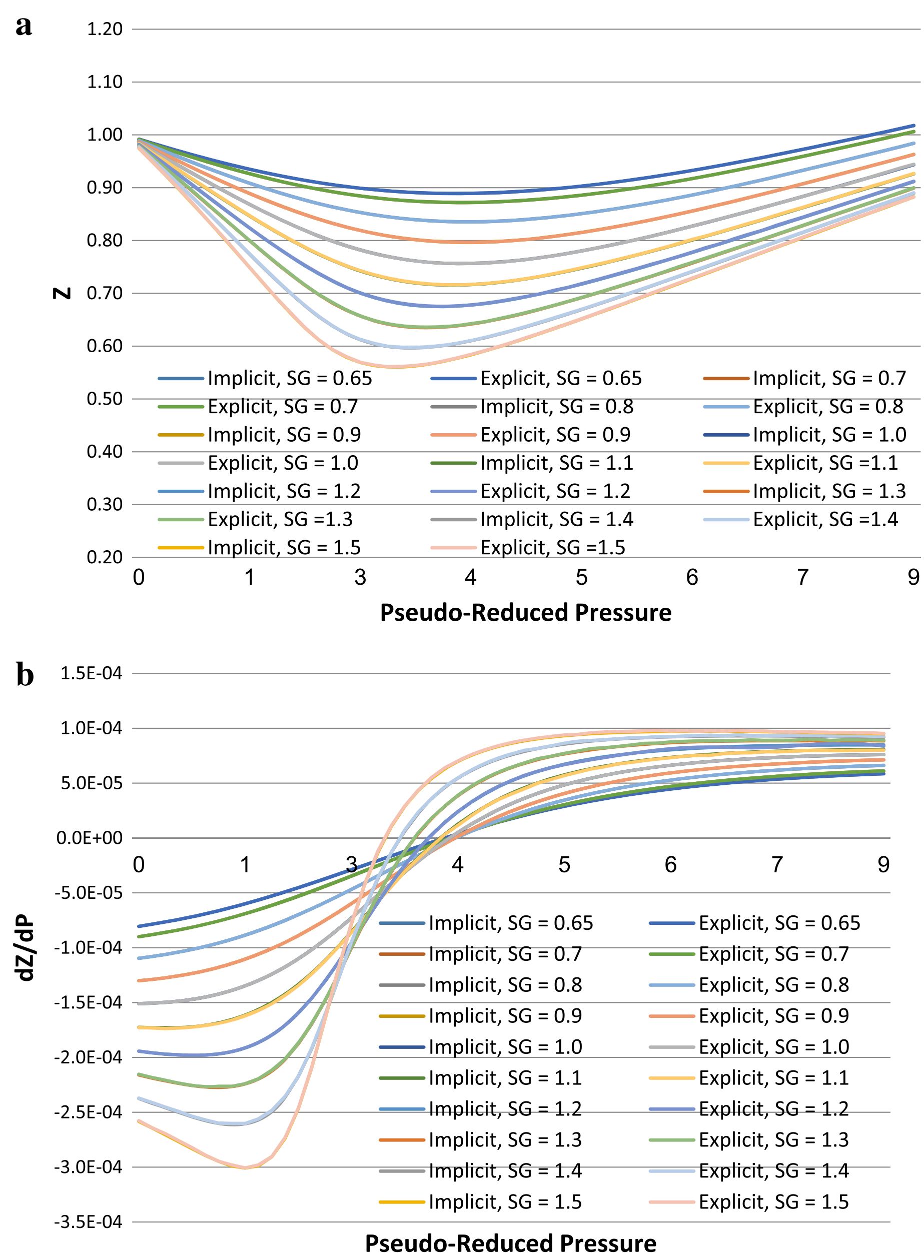

Fig. 3 **a** Plot of z versus pseudo-reduced pressure for nonlinear and ADM solutions, SG = 1.1−1.5, ADM 20-polynomials. **b** Plot of dzdp versus pseudo-reduced pressure for nonlinear and ADM solutions, SG = 1.1−1.5, ADM 20-polynomials

The series components for the Dranchuk–Abu-Kassem correlation are defined by:

$$\begin{aligned} z_0 &= 1 \\ z_{i+1} &= x_0 x_4 \mathrm{A}_{(i,1)} + x_1 x_4^2 \mathrm{A}_{(i,2)} + x_2 x_4^5 \mathrm{A}_{(i,3)} \\ &\quad + x_3 x_4^2 \mathrm{A}_{(i,4)} + a_{10} x_3 x_4^4 \mathrm{A}_{(i,5)}, \quad i \geq 0 \end{aligned} \tag{8}$$

$\mathrm{A}_{(i,1)}$, $\mathrm{A}_{(i,2)}$, $\mathrm{A}_{(i,3)}$, $\mathrm{A}_{(i,4)}$, and $\mathrm{A}_{(i,5)}$ are the Adomian polynomials representing the nonlinear terms as follows:

$$\begin{aligned} \sum_{i=0}^{\infty} \mathrm{A}_{(i,1)} &= \frac{1}{z} \\ \sum_{i=0}^{\infty} \mathrm{A}_{(i,2)} &= \frac{1}{z^2} \\ \sum_{i=0}^{\infty} \mathrm{A}_{(i,3)} &= \frac{1}{z^5} \\ \sum_{i=0}^{\infty} \mathrm{A}_{(i,4)} &= \frac{1}{z^2} \exp\left(\frac{-a_{10} x_4^2}{z^2}\right) \\ \sum_{i=0}^{\infty} \mathrm{A}_{(i,5)} &= \frac{1}{z^4} \exp\left(\frac{-a_{10} x_4^2}{z^2}\right) \end{aligned} \tag{9}$$

Similarly, the series components for the Dranchuk–Purvis–Robinson correlation are defined by:

Table 11 The implicit and ADM 10-polynomial solutions for gas gravities 1.1, 1.3, and 1.5

Gas gravity = 1.1		Gas gravity = 1.3		Gas gravity = 1.5	
Implicit	ADM	Implicit	ADM	Implicit	ADM
9.8276468E−01	9.8276468E−01	9.7854263E−01	9.7854263E−01	9.7445440E−01	9.7445440E−01
9.6548014E−01	9.6548014E−01	9.5680518E−01	9.5680518E−01	9.4833013E−01	9.4833013E−01
9.4817419E−01	9.4817419E−01	9.3479993E−01	9.3479993E−01	9.2160160E−01	9.2160160E−01
9.3088160E−01	9.3088161E−01	9.1254812E−01	9.1254813E−01	8.9424849E−01	8.9424853E−01
9.1364543E−01	9.1364545E−01	8.9008328E−01	8.9008334E−01	8.6626074E−01	8.6626103E−01
8.9651852E−01	8.9651858E−01	8.6745571E−01	8.6745596E−01	8.3764669E−01	8.3764840E−01
8.7956500E−01	8.7956514E−01	8.4473825E−01	8.4473916E−01	8.0844630E−01	8.0845356E−01
8.6286176E−01	8.6286208E−01	8.2203362E−01	8.2203628E−01	7.7875211E−01	7.7877572E−01
8.4649959E−01	8.4650021E−01	7.9948289E−01	7.9948919E−01	7.4874170E−01	7.4880184E−01
8.3058369E−01	8.3058473E−01	7.7727463E−01	7.7728651E−01	7.1872414E−01	7.1884408E−01
8.1523310E−01	8.1523452E−01	7.5565239E−01	7.5566932E−01	6.8919774E−01	6.8937532E−01
8.0057862E−01	8.0058003E−01	7.3491704E−01	7.3493157E−01	6.6089886E−01	6.6104956E−01
7.8675874E−01	7.8675924E−01	7.1541847E−01	7.1541194E−01	6.3479139E−01	6.3469141E−01
7.7391331E−01	7.7391162E−01	6.9753152E−01	6.9747502E−01	6.1192840E−01	6.1124074E−01
7.6217531E−01	7.6217035E−01	6.8161463E−01	6.8148116E−01	5.9317840E−01	5.9164626E−01
7.5166143E−01	7.5165345E−01	6.6795867E−01	6.6774744E−01	5.7895502E−01	5.7671527E−01
7.4246306E−01	7.4245477E−01	6.5674171E−01	6.5650503E−01	5.6914224E−01	5.6694167E−01
7.3463909E−01	7.3463623E−01	6.4800692E−01	6.4786078E−01	5.6324271E−01	5.6234827E−01
7.2821230E−01	7.2822260E−01	6.4167064E−01	6.4177187E−01	5.6060182E−01	5.6238607E−01
7.2316961E−01	7.2319998E−01	6.3755384E−01	6.3804154E−01	5.6057448E−01	5.6593057E−01
7.1946613E−01	7.1951840E−01	6.3542216E−01	6.3634066E−01	5.6260411E−01	5.7140191E−01
7.1703166E−01	7.1709867E−01	6.3502196E−01	6.3625525E−01	5.6624160E−01	5.7701290E−01
7.1577839E−01	7.1584219E−01	6.3610580E−01	6.3735413E−01	5.7113608E−01	5.8111898E−01
7.1560833E−01	7.1564241E−01	6.3844722E−01	6.3926571E−01	5.7701702E−01	5.8261275E−01
7.1641972E−01	7.1639571E−01	6.4184713E−01	6.4174839E−01	5.8367646E−01	5.8127862E−01
7.1811196E−01	7.1800972E−01	6.4613487E−01	6.4473795E−01	5.9095398E−01	5.7801031E−01
7.2058903E−01	7.2040749E−01	6.5116640E−01	6.4835683E−01	5.9872509E−01	5.7480243E−01
7.2376161E−01	7.2352661E−01	6.5682124E−01	6.5287651E−01	6.0689240E−01	5.7446207E−01
7.2754812E−01	7.2731373E−01	6.6299905E−01	6.5863365E−01	6.1537911E−01	5.8004360E−01
7.3187513E−01	7.3171582E−01	6.6961639E−01	6.6591242E−01	6.2412413E−01	5.9408037E−01
7.3667713E−01	7.3667076E−01	6.7660383E−01	6.7481652E−01	6.3307849E−01	6.1775490E−01
7.4189611E−01	7.4210048E−01	6.8390347E−01	6.8516270E−01	6.4220259E−01	6.5019790E−01
7.4748099E−01	7.4790919E−01	6.9146692E−01	6.9642877E−01	6.5146421E−01	6.8812126E−01
7.5338686E−01	7.5398908E−01	6.9925358E−01	7.0778309E−01	6.6083696E−01	7.2596103E−01
7.5957435E−01	7.6023333E−01	7.0722929E−01	7.1820791E−01	6.7029904E−01	7.5662904E−01
7.6600900E−01	7.6655452E−01	7.1536516E−01	7.2670799E−01	6.7983240E−01	7.7285145E−01
7.7266058E−01	7.7290436E−01	7.2363672E−01	7.3257201E−01	6.8942200E−01	7.6892477E−01
7.7950263E−01	7.7928882E−01	7.3202312E−01	7.3563198E−01	6.9905521E−01	7.4256995E−01
7.8651195E−01	7.8577286E−01	7.4050655E−01	7.3645171E−01	7.0872143E−01	6.9644973E−01
7.9366814E−01	7.9246977E−01	7.4907175E−01	7.3637411E−01	7.1841171E−01	6.3887078E−01
8.0095329E−01	7.9951389E−01	7.5770560E−01	7.3737378E−01	7.2811846E−01	5.8325499E−01
8.0835163E−01	8.0701942E−01	7.6639678E−01	7.4169702E−01	7.3783523E−01	5.4615076E−01
8.1584922E−01	8.1503360E−01	7.7513548E−01	7.5132322E−01	7.4755653E−01	5.4385935E−01
8.2343379E−01	8.2349688E−01	7.8391322E−01	7.6734091E−01	7.5727767E−01	5.8813514E−01
8.3109445E−01	8.3222493E−01	7.9272259E−01	7.8938670E−01	7.6699463E−01	6.8181094E−01

Table 11 Continued

Gas gravity = 1.1		Gas gravity = 1.3		Gas gravity = 1.5	
Implicit	ADM	Implicit	ADM	Implicit	ADM
8.3882157E−01	8.4092588E−01	8.0155716E−01	8.1532691E−01	7.7670397E−01	8.1550210E−01
8.4660657E−01	8.4925924E−01	8.1041129E−01	8.4135504E−01	7.8640277E−01	9.6664260E−01
8.5444186E−01	8.5693212E−01	8.1928007E−01	8.6261721E−01	7.9608847E−01	1.1018981E+00
8.6232063E−01	8.6381302E−01	8.2815916E−01	8.7436047E−01	8.0575892E−01	1.1834198E+00
8.7023686E−01	8.7002907E−01	8.3704477E−01	8.7343378E−01	8.1541225E−01	1.1784569E+00
8.7818513E−01	8.7600354E−01	8.4593357E−01	8.5979040E−01	8.2504687E−01	1.0706485E+00
8.8616061E−01	8.8239288E−01	8.5482261E−01	8.3749137E−01	8.3466141E−01	8.7011869E−01
8.9415896E−01	8.8990329E−01	8.6370930E−01	8.1466088E−01	8.4425471E−01	6.1866313E−01
9.0217631E−01	8.9900767E−01	8.7259137E−01	8.0196509E−01	8.5382578E−01	3.8630560E−01
9.1020916E−01	9.0964036E−01	8.8146678E−01	8.0952974E−01	8.6337377E−01	2.5676628E−01
9.1825438E−01	9.2100553E−01	8.9033376E−01	8.4278465E−01	8.7289796E−01	3.0219889E−01
9.2630915E−01	9.3166607E−01	8.9919073E−01	8.9844249E−01	8.8239777E−01	5.5183853E−01
9.3437092E−01	9.4004292E−01	9.0803631E−01	9.6247855E−01	8.9187270E−01	9.6423113E−01
9.4243741E−01	9.4530553E−01	9.1686926E−01	1.0122261E+00	9.0132235E−01	1.4169178E+00
9.5050654E−01	9.4834676E−01	9.2568851E−01	1.0240586E+00	9.1074636E−01	1.7282877E+00

Their plots are depicted in Fig. 2a

$$
\begin{aligned}
z_0 &= 1\\
z_{i+1} &= x_0 x_4 \mathrm{A}_{(i,1)} + x_1 x_4^2 \mathrm{A}_{(i,2)} + x_2 x_4^5 \mathrm{A}_{(i,3)}\\
&\quad + x_3 x_4^2 \mathrm{A}_{(i,4)} + a_7 x_3 x_4^4 \mathrm{A}_{(i,5)}, \quad i \geq 0 \qquad (10)
\end{aligned}
$$

$\mathrm{A}_{(i,1)}$, $\mathrm{A}_{(i,2)}$, $\mathrm{A}_{(i,3)}$, $\mathrm{A}_{(i,4)}$, and $\mathrm{A}_{(i,5)}$ are the Adomian polynomials representing the nonlinear terms as follows:

$$
\begin{aligned}
\sum_{i=0}^{\infty} \mathrm{A}_{(i,1)} &= \frac{1}{z}\\
\sum_{i=0}^{\infty} \mathrm{A}_{(i,2)} &= \frac{1}{z^2}\\
\sum_{i=0}^{\infty} \mathrm{A}_{(i,3)} &= \frac{1}{z^5}\\
\sum_{i=0}^{\infty} \mathrm{A}_{(i,4)} &= \frac{1}{z^2} \exp\left(\frac{-a_7 x_4^2}{z^2}\right)\\
\sum_{i=0}^{\infty} \mathrm{A}_{(i,5)} &= \frac{1}{z^4} \exp\left(\frac{-a_7 x_4^2}{z^2}\right) \qquad (11)
\end{aligned}
$$

Finally, the series components for the Hankinson–Thomas–Phillips correlation are defined by:

$$
\begin{aligned}
z_0 &= 1\\
z_{i+1} &= x_0 x_3 \mathrm{A}_{(i,1)} + x_1 x_3^2 \mathrm{A}_{(i,2)} + x_2 x_3^5 \mathrm{A}_{(i,3)}\\
&\quad + a_7 x_2 x_3^7 \mathrm{A}_{(i,4)}, \quad i \geq 0 \qquad (12)
\end{aligned}
$$

$\mathrm{A}_{(i,1)}$, $\mathrm{A}_{(i,2)}$, $\mathrm{A}_{(i,3)}$, and $\mathrm{A}_{(i,4)}$ are the Adomian polynomials representing the nonlinear terms as follows:

$$
\begin{aligned}
\sum_{i=0}^{\infty} \mathrm{A}_{(i,1)} &= \frac{1}{z}\\
\sum_{i=0}^{\infty} \mathrm{A}_{(i,2)} &= \frac{1}{z^2}\\
\sum_{i=0}^{\infty} \mathrm{A}_{(i,3)} &= \frac{1}{z^5} \exp\left(\frac{-a_7 x_3^2}{z^2}\right)\\
\sum_{i=0}^{\infty} \mathrm{A}_{(i,4)} &= \frac{1}{z^7} \exp\left(\frac{-a_7 x_3^2}{z^2}\right) \qquad (13)
\end{aligned}
$$

"Appendix B" is a listing of an efficient C# code of the *Adomian Decomposition Method (ADM)*. The first ten Adomian polynomials are calculated as given in Table 4, and the nonlinearities and their derivatives are calculated as given in Tables 5, 6, 7 for the Dranchuk–Abu-Kassem and Dranchuk–Purvis–Robinson correlations and are calculated as given in Tables 8, 9, 10 for the Hankinson–Thomas–Phillips correlation.

7 Adomian Polynomials

The *Adomian Decomposition Method (ADM)* is simple but the difficulty arises in the evaluation process of the *Adomian Polynomials*. The Adomian polynomials A_i can be calculated in a straightforward manner as follows:

Table 12 The implicit and ADM 20-polynomial solutions for gas gravities 1.1, 1.3, and 1.5

Gas gravity = 1.1		Gas gravity = 1.3		Gas gravity = 1.5	
Implicit	ADM	Implicit	ADM	Implicit	ADM
9.8276468E−01	9.8282253E−01	9.7854263E−01	9.7910085E−01	9.7445440E−01	9.7456242E−01
9.6548014E−01	9.6559491E−01	9.5680518E−01	9.5719036E−01	9.4833013E−01	9.4892108E−01
9.4817419E−01	9.4829876E−01	9.3479993E−01	9.3553096E−01	9.2160160E−01	9.2168975E−01
9.3088160E−01	9.3160618E−01	9.1254812E−01	9.1303098E−01	8.9424849E−01	8.9482312E−01
9.1364543E−01	9.1449150E−01	8.9008328E−01	8.9027620E−01	8.6626074E−01	8.6685924E−01
8.9651852E−01	8.9741710E−01	8.6745571E−01	8.6821040E−01	8.3764669E−01	8.3771455E−01
8.7956500E−01	8.7995613E−01	8.4473825E−01	8.4563277E−01	8.0844630E−01	8.0862911E−01
8.6286176E−01	8.6371589E−01	8.2203362E−01	8.2279802E−01	7.7875211E−01	7.7900547E−01
8.4649959E−01	8.4698078E−01	7.9948289E−01	7.9996224E−01	7.4874170E−01	7.4902592E−01
8.3058369E−01	8.3073099E−01	7.7727463E−01	7.7822266E−01	7.1872414E−01	7.1906679E−01
8.1523310E−01	8.1542921E−01	7.5565239E−01	7.5631082E−01	6.8919774E−01	6.8922227E−01
8.0057862E−01	8.0131783E−01	7.3491704E−01	7.3493105E−01	6.6089886E−01	6.6127774E−01
7.8675874E−01	7.8730092E−01	7.1541847E−01	7.1619500E−01	6.3479139E−01	6.3542541E−01
7.7391331E−01	7.7395629E−01	6.9753152E−01	6.9765169E−01	6.1192840E−01	6.1213575E−01
7.6217531E−01	7.6313620E−01	6.8161463E−01	6.8212732E−01	5.9317840E−01	5.9406002E−01
7.5166143E−01	7.5198896E−01	6.6795867E−01	6.6892000E−01	5.7895502E−01	5.7920503E−01
7.4246306E−01	7.4312215E−01	6.5674171E−01	6.5725994E−01	5.6914224E−01	5.7010442E−01
7.3463909E−01	7.3490020E−01	6.4800692E−01	6.4893603E−01	5.6324271E−01	5.6382088E−01
7.2821230E−01	7.2891194E−01	6.4167064E−01	6.4259030E−01	5.6060182E−01	5.6096518E−01
7.2316961E−01	7.2351469E−01	6.3755384E−01	6.3760103E−01	5.6057448E−01	5.6080164E−01
7.1946613E−01	7.2028238E−01	6.3542216E−01	6.3629615E−01	5.6260411E−01	5.6318181E−01
7.1703166E−01	7.1768114E−01	6.3502196E−01	6.3581181E−01	5.6624160E−01	5.6641537E−01
7.1577839E−01	7.1616006E−01	6.3610580E−01	6.3625842E−01	5.7113608E−01	5.7137829E−01
7.1560833E−01	7.1624676E−01	6.3844722E−01	6.3944139E−01	5.7701702E−01	5.7772124E−01
7.1641972E−01	7.1705461E−01	6.4184713E−01	6.4276452E−01	5.8367646E−01	5.8460087E−01
7.1811196E−01	7.1817711E−01	6.4613487E−01	6.4663492E−01	5.9095398E−01	5.9167492E−01
7.2058903E−01	7.2072220E−01	6.5116640E−01	6.5128844E−01	5.9872509E−01	5.9883478E−01
7.2376161E−01	7.2422075E−01	6.5682124E−01	6.5684734E−01	6.0689240E−01	6.0731733E−01
7.2754812E−01	7.2814258E−01	6.6299905E−01	6.6327243E−01	6.1537911E−01	6.1546502E−01
7.3187513E−01	7.3193109E−01	6.6961639E−01	6.7051521E−01	6.2412413E−01	6.2417980E−01
7.3667713E−01	7.3746894E−01	6.7660383E−01	6.7736362E−01	6.3307849E−01	6.3342743E−01
7.4189611E−01	7.4216447E−01	6.8390347E−01	6.8427820E−01	6.4220259E−01	6.4257447E−01
7.4748099E−01	7.4771119E−01	6.9146692E−01	6.9230623E−01	6.5146421E−01	6.5225486E−01
7.5338686E−01	7.5343255E−01	6.9925358E−01	6.9960782E−01	6.6083696E−01	6.6097430E−01
7.5957435E−01	7.5992495E−01	7.0722929E−01	7.0805933E−01	6.7029904E−01	6.7117310E−01
7.6600900E−01	7.6656231E−01	7.1536516E−01	7.1580861E−01	6.7983240E−01	6.7983473E−01
7.7266058E−01	7.7282387E−01	7.2363672E−01	7.2370928E−01	6.8942200E−01	6.9008178E−01
7.7950263E−01	7.7954701E−01	7.3202312E−01	7.3284882E−01	6.9905521E−01	6.9944444E−01
7.8651195E−01	7.8746663E−01	7.4050655E−01	7.4064175E−01	7.0872143E−01	7.0893442E−01
7.9366814E−01	7.9437769E−01	7.4907175E−01	7.4997921E−01	7.1841171E−01	7.1878615E−01
8.0095329E−01	8.0150559E−01	7.5770560E−01	7.5817085E−01	7.2811846E−01	7.2895631E−01
8.0835163E−01	8.0932949E−01	7.6639678E−01	7.6722286E−01	7.3783523E−01	7.3852227E−01
8.1584922E−01	8.1640117E−01	7.7513548E−01	7.7550766E−01	7.4755653E−01	7.4838354E−01
8.2343379E−01	8.2417606E−01	7.8391322E−01	7.8444391E−01	7.5727767E−01	7.5748756E−01
8.3109445E−01	8.3136308E−01	7.9272259E−01	7.9315934E−01	7.6699463E−01	7.6720999E−01

Table 12 Continued

Gas gravity = 1.1		Gas gravity = 1.3		Gas gravity = 1.5	
Implicit	ADM	Implicit	ADM	Implicit	ADM
8.3882157E−01	8.3907850E−01	8.0155716E−01	8.0205202E−01	7.7670397E−01	7.7750662E−01
8.4660657E−01	8.4729687E−01	8.1041129E−01	8.1132485E−01	7.8640277E−01	7.8649826E−01
8.5444186E−01	8.5469904E−01	8.1928007E−01	8.1965762E−01	7.9608847E−01	7.9683549E−01
8.6232063E−01	8.6254501E−01	8.2815916E−01	8.2885500E−01	8.0575892E−01	8.0650412E−01
8.7023686E−01	8.7084571E−01	8.3704477E−01	8.3802834E−01	8.1541225E−01	8.1617273E−01
8.7818513E−01	8.7904369E−01	8.4593357E−01	8.4607861E−01	8.2504687E−01	8.2543233E−01
8.8616061E−01	8.8685529E−01	8.5482261E−01	8.5487597E−01	8.3466141E−01	8.3559895E−01
8.9415896E−01	8.9497683E−01	8.6370930E−01	8.6459721E−01	8.4425471E−01	8.4486765E−01
9.0217631E−01	9.0237334E−01	8.7259137E−01	8.7327528E−01	8.5382578E−01	8.5459859E−01
9.1020916E−01	9.1053530E−01	8.8146678E−01	8.8162998E−01	8.6337377E−01	8.6403804E−01
9.1825438E−01	9.1876453E−01	8.9033376E−01	8.9068179E−01	8.7289796E−01	8.7298022E−01
9.2630915E−01	9.2657119E−01	8.9919073E−01	8.9983909E−01	8.8239777E−01	8.8335105E−01
9.3437092E−01	9.3534802E−01	9.0803631E−01	9.0811322E−01	8.9187270E−01	8.9224914E−01
9.4243741E−01	9.4265920E−01	9.1686926E−01	9.1760177E−01	9.0132235E−01	9.0156121E−01
9.5050654E−01	9.5127048E−01	9.2568851E−01	9.2644623E−01	9.1074636E−01	9.1095180E−01

Their plots are depicted in Fig. 3a

$$\mathrm{A}_i = \mathrm{A}_i\left(u_0, u_1, \ldots, u_i\right) = \frac{1}{i!}\frac{d^i}{d\lambda^i} N\left(\sum_{k=0}^{\infty} u_k \lambda^k\right)\Bigg|_{\lambda=0} \quad (14)$$

In this paper, we present three published algorithms in the literature that we found excellent for the calculation of the Adomian polynomials. Those algorithms, however, require the use of some symbolic software such as Maple, MATLAB, and Mathematica. Biazar et al. [8] introduced a generic algorithm for calculating the Adomian polynomials. The algorithm can be extended to calculate the Adomian polynomials for nonlinear functional with several variables, see "Appendix C". Biazar et al. [9] presented a Maple program to easily compute these polynomials for functions with one, two or several variables. This program also computes the Adomian polynomials for a system of functional equations as well. "Appendix D" lists this Maple program. Azreg–Aïnou [10] has presented a Mathematica program that efficiently generates the Adomian polynomials. "Appendix E" lists this Mathematica program for the 7th Adomian polynomial. For ease of access, however, we regenerated ten Adomian polynomials $\mathrm{A}_0-\mathrm{A}_{10}$ in a very compact form as it appears in Table 4. Tables 5, 6, 7 lists the required nonlinearities and their derivatives for the Dranchuk–Abu-Kassem and Dranchuk–Purvis–Robinson correlations. Tables 8, 9, 10 lists the required nonlinearities and their derivatives for the Hankinson–Thomas–Phillips correlation. The nonlinearities and their derivatives are written in a very efficient and compact form to simplify generation of higher order derivatives if more Adomian polynomials are required.

8 Results and Discussion

To check the validity of both nonlinear and ADM solutions of the above correlations, we solved the correlations for gas gravities between 0.65 and 1.5 at a step size of 1.0 and for high-pressure values above 6,000 psi at a step size of 100 psi. Figure 1a depicts the plots of the gas compressibility factor, z, versus the pseudo-reduced pressure for both nonlinear and ADM solutions (using 10 ADM polynomials) for gas gravities 0.65–1.0. On the other hand, Fig. 1b depicts the plots of dzdp (change of z with pressure) versus pseudo-reduced pressure for both solutions for gas gravities 0.65–1.0. Both Figures indicate that the ADM solution matches the nonlinear in an excellent manner. Figure 1a, however, reveals minute oscillations of the ADM solution around the nonlinear solution for gas gravity of 1.0.

Figure 2a depicts the plots of the gas compressibility factor, z, versus the pseudo-reduced pressure for both nonlinear and ADM solutions (using 10 ADM polynomials) for gas gravities 1.1–1.5. On the other hand, Fig. 2b depicts the plots of dzdp (change of z with pressure) versus pseudo-reduced pressure for both solutions for gas gravities 1.1–1.5. It is clear from Fig. 2a that the ADM solution starts oscillating as gas gravity and pressure increase. The same oscillation happens for dzdp as depicted by Fig. 2b. This oscillation behavior suggests adding more Adomian polynomials for the ADM solution to match the nonlinear one. Said differently, we may need up to 20 Adomian polynomials at a minimum for the ADM solution to match the nonlinear one. Figure 3a depicts the plots of the gas compressibility factor, z, versus the pseudo-reduced pressure for both nonlinear and ADM solutions (using 20 ADM polynomials) for gas gravities 1.1–1.5. On the other hand, Fig. 3b depicts the plots of $\mathrm{d}z\mathrm{d}p$ (change of z with pressure) versus pseudo-reduced pressure for both solutions for gas gravities 1.1–1.5. It is clear from Fig. 3a that the ADM solution matches the nonlinear in an excellent manner. For completeness, however, both implicit and ADM 10- and 20-polynomial solutions for gas gravities 1.1, 1.3, and 1.5 are listed in Tables 11 and 12, respectively. Similarly Fig. 3b depicts an excellent match with the nonlinear solution. In addition, it is clear from Table 4 that the manual generation process of the Adomian polynomials is a bit lengthy and cumbersome. It is simplified via the use of some symbolic software such as Maple, MATLAB, and Mathematica. The manual generation of the ADM polynomials presented in Table 4 is given for ease of access and manual calculations only.

9 Conclusions

1. The Newton–Raphson- Bisection nonlinear solution method proved to be efficient in almost all ranges of pressure, temperature, and gas gravities.
2. The *Adomian Decomposition Method (ADM)* is a reliable tool for solving nonlinear equations. The drawback of the method, however, stems from the difficulty of the Adomian polynomials generation process. It is a bit lengthy and requires some symbolic software such as Maple, MATLAB, and Mathematica.
3. For gas gravity ranges of 0.6–1.0, the ADM requires a minimum of ten Adomian polynomials to match the nonlinear solution. However, when gas gravity increases above 1.0, more Adomian polynomial are needed; a 15–20 range at a minimum for ADM to match the nonlinear solution.
4. To the best of our knowledge, this is the first research that implements the ADM method of solution to the above-listed z factor correlations. This research may be used as a starting point for more researches in using ADM to tackle nonlinear issues in other petroleum problems.
5. In this paper, we list the first ten Adomian polynomials along with their nonlinearities and derivatives, for the above-mentioned z factor correlations, in a more compact form that simplifies derivation of higher order terms.
6. It investigates the validity of the ADM and nonlinear solutions of the gas compressibility factors for gas gravity ranges of 0.6–1.5 at a step size of 0.1 and high-pressure values above 6,000 psi.

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Appendix A

An efficient C# code for the Newton–Raphson-Bisection nonlinear solver

```
public static double NewtonRaphsonBisection(double x1, double x2, double tol, out int iter,
                            FunctionPlusDerivative funcd)
{
    const int MAXIT = 1000;
    int j;
    double df, dx, dxold, f, fh, fl, temp, xh, xl, rts;
    iter = 0;

    funcd(x1, out fl, out df);
    funcd(x2, out fh, out df);
    if ((fl > 0.0 && fh > 0.0) || (fl < 0.0 && fh < 0.0))
        return 0.0; // nrerror("Root must be bracketed in rtsafe");
    if (fl == 0.0) return x1;
    if (fh == 0.0) return x2;
    if (fl < 0.0)
    {
        xl = x1;
        xh = x2;
    }
    else
    {
        xh = x1;
        xl = x2;
    }
    rts = 0.5 * (x1 + x2);
    dxold = System.Math.Abs(x2 - x1);
    dx = dxold;
    funcd(rts, out f, out df);
    for (j = 0; j < MAXIT; j++)
    {
        if ((((rts - xh) * df - f) * ((rts - xl) * df - f) > 0.0)
           || (System.Math.Abs(2.0 * f) > System.Math.Abs(dxold * df)))
    {
        dxold = dx;
        dx = 0.5 * (xh - xl);
        rts = xl + dx;
        if (xl == rts)
        {
            iter = j;
            return rts;
        }
    }
    else
    {
        dxold = dx;
        dx = f / df;
        temp = rts;
        rts -= dx;
        if (temp == rts)
        {
            iter = j;
            return rts;
        }
    }
    if (System.Math.Abs(dx) < tol)
    {
        iter = j;
        return rts;
    }

    funcd(rts, out f, out df);
    if (f < 0.0)
        xl = rts;
    else
        xh = rts;
    }
    return 0.0; // nrerror("Maximum number of iterations exceeded in rtsafe");
}
```

Appendix B

An efficient C# code for the Adomian Decomposition Method (ADM)

```
public override double GetZ(double P, out double dZdP)
{
    double Ppr = P / this.Ppc;
    if (Ppr >= 5.0) this.i = 1;
    this.x[this.i][3] = this.ct * Ppr;

    int nZ = 12;
    double[] z = new double[nZ];
    z[0] = 1.0;

    double[] f1 = this.GetF1(z[0]);
    double[] f2 = this.GetF2(z[0]);
    double[] f3 = this.GetF3(z[0]);
    double[] f4 = this.GetF4(z[0]);

    double[] A1 = new double[nZ - 1];
    double[] A2 = new double[nZ - 1];
    double[] A3 = new double[nZ - 1];
    double[] A4 = new double[nZ - 1];

    double X1 = this.t1 * this.x[this.i][0] * this.x[this.i][3];
    double X2 = this.t1 * this.x[this.i][1] * System.Math.Pow(this.x[this.i][3], 2);
    double X3 = this.t1 * this.x[this.i][2] * System.Math.Pow(this.x[this.i][3], 5);
    double X4 = this.a[this.i][7] * this.t1 * this.x[this.i][2] * System.Math.Pow(this.x[this.i][3], 7);

    for (int i = 0; i < nZ - 1; i++)
    {
        A1[i] = this.GetAdomianPolynomial(i, z, f1);
        A2[i] = this.GetAdomianPolynomial(i, z, f2);
        A3[i] = this.GetAdomianPolynomial(i, z, f3);
        A4[i] = this.GetAdomianPolynomial(i, z, f4);
        z[i + 1] = X1 * A1[i] + X2 * A2[i] + X3 * A3[i] + X4 * A4[i];
    }

    double Z = 0.0;
    for (int i = 0; i < z.Length; i++)
        Z += z[i];

    this.r1 = this.x[this.i][3] / Z;
    this.r2 = this.r1 * this.r1;
    this.r5 = this.r1 * this.r2 * this.r2;
    this.cr = this.a[this.i][7] * this.r2;
    double fp = this.x[this.i][0] * this.t1 + 2.0 * this.x[this.i][1] * this.t1 * this.r1
                + 2.0 * this.x[this.i][2] * this.t1 * this.r2 * this.r2 * (2.5 + this.cr *
                (2.5 - this.cr)) * System.Math.Exp(-this.cr) + Z / this.r1;

    double c1 = this.x[this.i][3] * (this.x[this.i][3] / this.r2 - fp) / Z;
    dZdP = c1 / (P * (c1 / Z - 1.0));
    return Z;
}
```

Appendix C

Biazar et al. [8] generic algorithm to generate the Adomian polynomials $A_0, A_1, \ldots, A_n$

Step 1: Input the nonlinear term $G(u)$ and n , the number of Adomian polynomials needed.
Step 2: Set $A_0 = G(u_0)$
Step 3: For $k = 0$ to $n - 1$ do

$A_k(u_0, \ldots\ldots, u_k) := A_k(u_0 + u_1\lambda, \ldots\ldots, u_k + (k+1)u_{k+1}\lambda)$
$\{\text{in } A_k : u_i \to u_i + (i+1)u_{i+1}\lambda \quad \text{for } i = 0, 1, \ldots, k\}$

Step 4: Taking the first order derivative of A_k , with respect to λ, and then let $\lambda = 0$:

$$\frac{d}{d\lambda}A_k\bigg|_{\lambda=0} = (k+1)A_{k+1}$$

End do
Step 5: Output $A_0, A_1, \ldots, A_n$.

Appendix D

Biazar et al. [9] Maple program to generate the Adomian polynomials $A_0, A_1, \ldots, A_n$

```
>restart;
>with(student):
INPUT
>G[1]:=G(u[1],u[2],...,u[N]);
G[2]:=G(u[1],u[2],...,u[N]);
...
G[k]:=G(u[1],u[2],...,u[N]);
>m:=M:
>n:=N:
>k:=K:
PROGRAM
> for i from 1 by 1 while i <= n do
u[i,lambda]:=sum(u[i,b]*(lambda)_b,b=0..m):
end do:
>for i from 1 by 1 while i <= n
do for j from 1 by 1 while j <= n
do G[i,lambda]:=subs(u[j](t)=u[j,lambda],G[i]):
G[i]:=G[i,lambda]
end do:
end do;
>for i from 1 by 1 while i <= n do
s[i]:=expand(G[i,lambda],lambda):
ft[i]:= unapply(s[i],lambda):
end do:
>for j from 1 by 1 while j <= k
do for i from 0 by 1 while i <= m
do A[j][i]:=((D@@i)(ft[j])(0)/i!):
print( A[j,i],"=", A[j][i])
end do;
end do;
```

Appendix E

Azreg–Aïnou [10] Mathematica program to generate the Adomian polynomials $A_0, A_1, \ldots, A_n$ for $m = 7$

$\mathrm{m = 7; P = Table}\left[0, \{j, m\}\right];$
$\mathrm{If[m > 1}, k = 0;$
$\mathrm{While[(k = k + 1) \le m,}$

$$\mathrm{P[[k]] = Reduce}\left[\sum_{i=1}^{m} n_i == k\&\&\sum_{i=1}^{m} i n_i\right.$$
$$== m\&\&\sum_{i=1}^{m}\mathrm{Abs}[n_i] == \sum_{i=1}^{m} n_i\&\&\mathrm{Table}[n_i, \{i, m\}]$$
$$\left.\in \mathrm{Integers}]\right];$$

$\mathrm{U = P/.\{n \to u\};}$
$A_{m,1} = 0; A_{m,2} = 0;$
$\mathrm{Do[If[Length[U[[k, 1, 1, 1]]] == 0,}$
$$A_{m,1} = A_{m,1} + \frac{\prod_{j=1}^{m} \mathrm{U[[k,j,1]]}^{\mathrm{P[[k,j,2]]}}}{\prod_{j=1}^{m}(\mathrm{P[[k,j,2]]!})}\mathrm{Derivative[k][F][u_0]},$$
$$A_{m,2} = A_{m,2} + \left(\sum_{i=1}^{\mathrm{Length[P[[k]]]}} \frac{\prod_{j=1}^{m} \mathrm{U[[k,i,j,1]]}^{\mathrm{P[[k,i,j,2]]}}}{\prod_{j=1}^{m}(\mathrm{P[[k,i,j,2]]!})}\right)\mathrm{Derivative}$$
$$\mathrm{[k][F][u_0]},$$
$\mathrm{\{k, 1, m\}];}$
$A_m = A_{m,1} + A_{m,2}\}$
$\mathrm{If[m == 1}, A_m = u_1 \mathrm{Derivative[1][F][u_0]]}$
$\mathrm{If[m == 0}, A_m = \mathrm{F[u_0]]}$

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